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HIGH-ORDER BEM FORMULATIONS FOR STRONGLY NON-LINEAR PROBLEMS GOVERNED BY QUITE GENERAL NON-LINEAR DIFFERENTIAL OPERATORS. PART 2: SOME 2D EXAMPLES

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SUMMARY

In this paper the general BEM proposed previously by Liao is applied to solve some 2D strongly non-linear differential equations, even including those whose governing equations and boundary conditions do not contain any linear terms. It is shown that the proposed general BEM is really valid for general non-linear problems, so that it can be applied to solve high-dimensional, strongly non-linear problems in engineering. \odot 1997 by John Wiley & Sons, Ltd. Int. j. numer. methods fluids 24: 863–873, 1997.

(No. of Figures: 8. No. of Tables: 1. No. of Refs: 13.)

KEY WORDS: general BEM; general non-linear differential operator; homotopy

1. INTRODUCTION

The boundary element method¹⁻⁴ (BEM) is in principle based on the linear superposition of the fundamental solution of a linear operator. Nowadays, many researchers^{5–6} apply the BEM to solve non-linear problems. The basic idea of the current BEM for nonlinear problems is to move all nonlinear terms to the right-hand side of the equations and then find the corresponding fundamental solutions of the linear operator remaining on the left-hand side of the equations. In the BEM for nonlinear problems, iteration is necessary and a domain integral term appears.

The above-mentioned BEM for non-linear differential equations has some obvious restrictions. First of all, it is invalid if nothing is left after moving all non-linear terms of an equation to its righthand side, i.e. the equation does not contain any linear terms so that there certainly does not exist a fundamental solution at all. Secondly, even if a linear operator exists, it may be so simple that it can not satisfy all boundary conditions. Finally, this linear operator might be so complex that its fundamental solution is unknown or quite difficult to find. In the first two cases the traditional BEM for non-linear problems does not work at all. In the last case it is not easy to apply the BEM.

Liao^{7,8} proposed a new kind of BEM for quite general non-linear differential equations even including those whose governing equations and boundary conditions do not contain any linear terms at all. This general BEM can overcome the three above-mentioned restrictions of the traditional BEM for non-linear problems. It is based on homotopy in topology so that it has a solid mathematical base.

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This general BEM offers great freedom in selecting the linear operator and initial approximation. Some examples of *one-dimensional* highly non-linear differential equations are given in References 7 and 8. The high-order BEM formulaes for general governing equations and boundary conditions are given in Reference 8.

This paper is the continuation of the author's work described in Reference 8. In this paper the general BEM proposed by Liao in References 7 and 8 is further applied to solve some 2D strongly non-linear problems whose governing equation and boundary conditions do not contain any linear terms at all. The purpose of this paper is to show that the proposed general BEM is really valid for *high-dimensional* strongly non-linear differential equations.

2. BASIC IDEAS OF THE PROPOSED GENERAL BEM

Consider the non-linear differential equation

$$
A(u) = f(\vec{r}), \quad \vec{r} \in \Omega,
$$
\n⁽¹⁾

with boundary conditions

$$
H\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad \vec{r} \in \Gamma,
$$
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(*u*) = $f(\vec{r})$, $\vec{r} \in \Omega$, (1)

(*u*, $\frac{\partial u}{\partial n}$) = 0, $\vec{r} \in \Gamma$, (2)
 r, $f(\vec{r})$ is a known function of the co-ordinates of the point

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 $u(v) - f(\$ \vec{r} s:
 \hat{s} simplicity, we define $u' = \partial u / \partial n$ (on the boundary Γ) in this paper.

 $\frac{1}{2}$ satisfies

$$
(1 - p)[L(v) - L(u_0)] + p[A(v) - f(\vec{r})] = 0, \quad p \in [0, 1], \quad \vec{r} \in \Omega,
$$
\n(3)

with boundary condition

$$
H(v, v') = (1 - p)H(u_0, u'_0), \quad p \in [0, 1], \quad \vec{r} \in \Gamma,
$$
\n(4)

we obtain a *family* of iterative BEM formulae at high order,

\n- $$
\in \Omega
$$
 and *H* is a function of *u* and its derivatives $\frac{\partial u}{\partial n}$ on the boundary Γ of the domain Ω . For simplicity, we define $u' = \frac{\partial u}{\partial n}$ (on the boundary Γ) in this paper.
\n- In References 7 and 8, by means of constructing a homotopy⁹ $v(\vec{r}, p) : \Omega \times [0, 1] \rightarrow \mathbb{R}$ which it is $(1 - p)[L(v) - L(u_0)] + p[A(v) - f(\vec{r})] = 0$, $p \in [0, 1]$, $\vec{r} \in \Omega$, (3) $H(v, v') = (1 - p)H(u_0, u'_0)$, $p \in [0, 1]$, $\vec{r} \in \Gamma$, (4) $u_{k+1}(\vec{r}) = u_k(\vec{r}) + \sum_{m=1}^{M} \left(\frac{v_0^{[m]}(\vec{r})}{m!} \right) \lambda^m$ $(k = 0, 1, 2, 3, \ldots)$, (5) *here L* is a properly selected *linear* operator whose fundamental solution is known, $u_0(\vec{r})$ is an initial approximation which can be selected with great freedom, $p \in [0, 1]$ is an imbedding parameter and
\n

In References 7 and 8, by means of constructing a homotopy⁹ $v(\vec{r}, p)$
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 $(1-p)[L(v) - L(u_0)] + p[A(v) - f(\vec{r})] = 0, \quad p \in [0, 1],$
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 $H(v, v') = (1-p)H(u_0, u'_0), \quad p \in [0, 1], \quad \vec{r} \in \Gamma,$
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 nation derivative), $p \in [0, 1], \quad \vec{r} \in \Gamma$, (4)

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 $(2, 3, \cdots)$, (7)
 \circledcirc 1997 by John Wiley & Sons, Ltd where *L* is a properly selected *linear* operator whose fundamental solution is known, $u_0(\vec{r})$ is an initial (\vec{r}) is an initial
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led $mth\text{-}order$
(6) approximation which can be selected with great freedom, $p \in [0, 1]$ is an imbedding parameter and $v(\vec{r}, p)$ is now a function of both $\vec{r} \in \Omega$ and $p \in [0, 1]$. The term

$$
v_0^{[m]}(\vec{r}) = \frac{\partial^m v(\vec{r}, p)}{\partial p^m}\Big|_{p=0}
$$

derivative and is de

$$
v_0(\vec{r}), \quad \vec{r} \in \Omega \quad (m = 1)
$$

$$
\frac{\partial v_0^{[m]}(\vec{r})}{\partial \vec{r}} = h_m(\vec{r}), \quad \vec{r}
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 $v(\vec{r}, p)$ is now a function of both $\vec{r} \in \Omega$ and $p \in [0, 1]$. The term
 $v_0^{[m]}(\vec{r}) = \frac{\partial^m v(\vec{r}, p)}{\partial p^m}\Big|_{p=0}$

is called the *mth-order deformation derivative* and is deter
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 $L(v_0^{[m]}) = f_m(\$ ∂p^m
and
 Ω is called the *mth-order deformation derivative* and is determined by the so-called *mth-order deformation equation*

$$
L(v_0^{[m]}) = f_m(\vec{r}), \quad \vec{r} \in \Omega \quad (m = 1, 2, 3, \cdots), \tag{6}
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with boundary condition

approximation which can be selected with great freedom,
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v_0^{[m]}(\vec{r}) = \frac{\partial^m v(\vec{r}, p)}{\partial p^m}\Big|_{p=0}
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$$
L(v_0^{[m]}) = f_m(\vec{r}), \quad \vec{r} \in \Omega \quad (m = 1, 2, 3, \cdots),
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\nwith boundary condition
\n
$$
\frac{\partial H}{\partial u}\Big|_{p=0} v_0^{[m]}(\vec{r}) + \frac{\partial H}{\partial u'}\Big|_{p=0} \frac{\partial v_0^{[m]}(\vec{r})}{\partial n} = h_m(\vec{r}), \quad \vec{r} \in \Gamma \quad (m = 1, 2, 3, \cdots),
$$
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where

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f_1(\vec{r}) = f(\vec{r}) - A(u_0),
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$$
f_1(\vec{r}) = f(\vec{r}) - A(u_0),
$$
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\n
$$
f_m(\vec{r}) = m \left(L(v_0^{[m-1]}) - \frac{d^{m-1}[A(v)]}{dp^{m-1}} \Big|_{p=0} \right) \quad (m > 1)
$$
\n(9)
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$$
h_1(\vec{r}) = -H(u_0, u'_0),
$$
\n(10)
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$$
= -\left[\frac{\partial^2 H}{\partial u^2} (v_0^{[1]})^2 + 2 \frac{\partial^2 H}{\partial u \partial u'} v_0^{[1]} \frac{\partial v_0^{[1]}}{\partial n} + \frac{\partial^2 H}{\partial (u')^2} \left(\frac{\partial v_0^{[1]}}{\partial n} \right)^2 \right] \Big|_{p=0},
$$
\n(11)

and

$$
h_1(\vec{r}) = -H(u_0, u'_0),\tag{10}
$$

$$
f_m(\vec{r}) = m \bigg(L(v_0^{[m-1]}) - \frac{d^{m-1}[A(v)]}{dp^{m-1}} \bigg|_{p=0} \bigg) \quad (m > 1)
$$
\nand\n
$$
h_1(\vec{r}) = -H(u_0, u'_0),
$$
\n
$$
h_2(\vec{r}) = -\left[\frac{\partial^2 H}{\partial u^2} (v_0^{[1]})^2 + 2 \frac{\partial^2 H}{\partial u \partial u'} v_0^{[1]} \frac{\partial v_0^{[1]}}{\partial n} + \frac{\partial^2 H}{\partial (u')^2} \left(\frac{\partial v_0^{[1]}}{\partial n} \right)^2 \right] \bigg|_{p=0},
$$
\n(Note that equation (6) is *linear* with *linear* boundary condition (7). Moreover, the linear operator *L* can be properly selected so that its fundamental solution is known. Therefore the *m*th-order deformation equation (6) under condition (7) can be easily solved by the traditional BEM in the

 $\overrightarrow{(r)} = m \left(L(v) \right)$
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 $+\frac{\partial^2 H}{\partial (u')^2} \left(\frac{\partial v_0^{[1]}}{\partial n}\right)^2$, (11)

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)]dΓ + $\int_{\Omega} f_m \omega d\Omega$, (12) $(\vec{r}) = -\left[\frac{\partial^2 H}{\partial u^2}\right]$

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 $\frac{1}{2}$ can be properly selected so that its fundamental solution is known. Therefore the *m*th-order deformation equation (6) under condition (7) can be easily solved by the traditional BEM in the following way:

$$
c(\vec{r})v_0^{[m]}(\vec{r}) = \int_{\Gamma} \left[v_0^{[m]}B(\omega) - \omega B(v_0^{[m]}) \right] d\Gamma + \int_{\Omega} f_m \omega d\Omega, \tag{12}
$$

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n Refer where *B* is the corresponding boundary operator for the freely selected linear operator *L*, ω is the fundamental solution of *L* and $c(\vec{r})$ is a known coefficient dependent upon the co-ordinates of the point \vec{r} . The detailed formulations of the above equations are given in Reference 7 and 8.

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ie that we n \vec{r} . The detailed formulations of the above equations are given in Reference 7 and 8.
er selecting an initial approximation $u_0(\vec{r})$, the term $f_m(\vec{r})$ on the right-hand side of equ
wn for each m ($m \ge 1$). Note After selecting an initial approximation *u*⁰(\vec{r}), the term $f_m(\vec{r})$ on the right-hand side of equation (6)
ow have very great freedom to select the corresponding
now select a proper linear operator *L* whose fundamental
rator *A* under consideration does *not* is known for each $m (m \geq 1)$. Note that we now have very great freedom to select the corresponding $(m \ge 1)$. Note that we now have very great freedom to select the corresponding
nore precisely, we can now select a proper linear operator *L* whose fundamental
en if the non-linear operator *A* under consideration does *n* linear operator *L*, or more precisely, we can now select a proper linear operator *L* whose fundamental solution is known even if the non-linear operator *A* under consideration does *not* contain any linear terms at all. In the special case where the operator *A* can be divided into two parts, one linear, the other non-linear, so that $A = \hat{L} + \hat{N}$ holds, and moreover, \hat{L} is proper and used as the linear operator becomes

other non-linear, so that
$$
A = L + N
$$
 holds, and moreover, L is proper and used as the linear operator whose fundamental solution is $\hat{\omega}$, then the above-mentioned general BEM formula in case $M = 1$ becomes
\n
$$
c(\vec{r})v_0^{[1]}(\vec{r}) = \int_{\Gamma} [v_0^{[1]} \hat{B}(\hat{\omega}) - \hat{\omega} \hat{B}v_0^{[1]}] \, \text{d}\Gamma + \int_{\Omega} [f - \hat{L}(u) - \hat{N}(u)] \hat{\omega} \, \text{d}\Omega
$$
\n
$$
= \int_{\Gamma} [v_0^{[1]} \hat{B}(\hat{\omega}) - \hat{\omega} \hat{B}(v_0^{[1]})] \, \text{d}\Gamma + \int_{\Omega} [f - \hat{N}(u)] \hat{\omega} \, \text{d}\Omega - c(\vec{r}) u(\vec{r}) + \int_{\Gamma} [u \hat{B}(\hat{\omega}) - \hat{\omega} \hat{B}(u)] \, \text{d}\Gamma,
$$
\nwhich gives
\n
$$
c(\vec{r}) [u(\vec{r}) + v_0^{[1]}(\vec{r})] = \int_{\Gamma} [(u + v_0^{[1]}) \hat{B}(\hat{\omega}) - \hat{\omega} \hat{B}(u + v_0^{[1]})] \, \text{d}\Gamma + \int_{\Omega} [f(\vec{r}) - \hat{N}(u)] \hat{\omega} \, \text{d}\Omega, \quad \vec{r} \in \Omega. \quad (14)
$$
\nLet $u_k(\vec{r}) = u(\vec{r})$ and $\tilde{u}_k(\vec{r}) = u(\vec{r}) + v_0^{[1]}(\vec{r})$. The above expression can be rewritten as
\n
$$
c(\vec{r}) \tilde{u}_k(\vec{r}) = \int_{\Gamma} [\tilde{u}_k \hat{B}(\hat{\omega}) - \hat{\omega} \hat{B}(\tilde{u}_k)] \, \text{d}\Gamma + \int_{\Omega} [f(\vec{r}) - \hat{N}(u_k)] \hat{\omega} \, \text{d}\Omega,
$$
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\n(1997)

which gives

$$
\int_{\Gamma} \left[(u + v_0)^{1/2} + \int_{\Omega} \left[(u + v_0^{[1]}) \hat{B}(\hat{\omega}) - \hat{\omega} \hat{B}(u + v_0^{[1]}) \right] d\Gamma + \int_{\Omega} \left[f(\vec{r}) - \hat{N}(u) \right] \hat{\omega} d\Omega, \quad \vec{r} \in \Omega. \quad (14)
$$
\n
$$
\text{et } u_k(\vec{r}) = u(\vec{r}) \text{ and } \tilde{u}_k(\vec{r}) = u(\vec{r}) + v_0^{[1]}(\vec{r}). \text{ The above expression can be rewritten as}
$$
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$$
c(\vec{r}) \tilde{u}_k(\vec{r}) = \int_{\Gamma} \left[\tilde{u}_k \hat{B}(\hat{\omega}) - \hat{\omega} \hat{B}(\tilde{u}_k) \right] d\Gamma + \int_{\Omega} \left[f(\vec{r}) - \hat{N}(u_k) \right] \hat{\omega} d\Omega, \quad (15)
$$
\n
$$
\text{or } \tilde{u}_k(\vec{r}) = \int_{\Gamma} \left[\tilde{u}_k \hat{B}(\hat{\omega}) - \hat{\omega} \hat{B}(\tilde{u}_k) \right] d\Gamma + \int_{\Omega} \left[f(\vec{r}) - \hat{N}(u_k) \right] \hat{\omega} d\Omega, \quad (15)
$$
\n
$$
\text{or } \tilde{u}_k(\vec{r}) = \int_{\Gamma} \left[\tilde{u}_k \hat{B}(\hat{\omega}) - \hat{\omega} \hat{B}(\tilde{u}_k) \right] d\Gamma + \int_{\Omega} \left[f(\vec{r}) - \hat{N}(u_k) \right] \hat{\omega} d\Omega, \quad (15)
$$
\n
$$
\text{or } \tilde{u}_k(\vec{r}) = \int_{\Gamma} \left[\tilde{u}_k \hat{B}(\hat{\omega}) - \hat{\omega} \hat{B}(\tilde{u}_k) \right] d\Gamma + \int_{\Omega} \left[f(\vec{r}) - \hat{N}(u_k) \right] \hat{\omega} d\Omega, \quad (15)
$$

Let $u_k(\vec{r}) = u(\vec{r})$ and $\tilde{u}_k(\vec{r}) = u(\vec{r}) + v_0^{11}(\vec{r})$. The above express $(\vec{r}) = u(\vec{r})$ and \tilde{u}
 $c(\vec{r})$
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$$
[v_0^{[1]}B(\hat{\omega}) - \hat{\omega}B(v_0^{[1]})]d\Gamma + \int_{\Omega} [f - N(u)]\hat{\omega}d\Omega - c(\vec{r})u(\vec{r}) + \int_{\Gamma} [uB(\hat{\omega}) - \hat{\omega}B(u)]d\Gamma,
$$
\n(13)
\n
$$
\vec{r})] = \int_{\Gamma} [(u + v_0^{[1]})\hat{B}(\hat{\omega}) - \hat{\omega}\hat{B}(u + v_0^{[1]})]d\Gamma + \int_{\Omega} [f(\vec{r}) - \hat{N}(u)]\hat{\omega}d\Omega, \quad \vec{r} \in \Omega.
$$
\n(14)
\nand $\tilde{u}_k(\vec{r}) = u(\vec{r}) + v_0^{[1]}(\vec{r}).$ The above expression can be rewritten as
\n
$$
c(\vec{r})\tilde{u}_k(\vec{r}) = \int_{\Gamma} [\tilde{u}_k\hat{B}(\hat{\omega}) - \hat{\omega}\hat{B}(\tilde{u}_k)]d\Gamma + \int_{\Omega} [f(\vec{r}) - \hat{N}(u_k)]\hat{\omega}d\Omega,
$$
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\text{INT. J. NUMBER. METHODS FLUIDS, VOL. 24: 863-873 (1997)}
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which is exactly the formula of the traditional BEM for non-linear problems. Moreover, if a relaxation parameter λ is introduced into the iteration, we have

$$
u_{k+1}(\vec{r}) = u_k(\vec{r}) + \lambda [\tilde{u}_k(\vec{r}) - u_k(\vec{r})]
$$

= $u_k(\vec{r}) + \lambda v_0^{[1]}(\vec{r}),$ (16)

which is exactly formula (5) in the case $M = 1!$ This means that the traditional boundary element method for non-linear problems is indeed only a special case of the newly proposed boundary element method, so that our above-mentioned BEM is more general.

 $u_{+1}(\vec{r}) = u_k(\vec{r}) + \lambda \tilde{u}$
= $u_k(\vec{r}) + \lambda v_0^{\dagger}$
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roblems $\tilde{u}_k(\vec{r}) - u_k(\vec{r})$
 $\begin{bmatrix} \tilde{u}_j(\vec{r}) & \tilde{v}_j(\vec{r}) \\ \tilde{v}_j(\vec{r}) & \tilde{v}_j(\vec{r}) \end{bmatrix}$, s means that pecial case of s more gener is to use the newly proposed in the orig to use the n s to use the n g non-lineari EXAMPL $= u_k(\vec{r}) + \lambda v$
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experator A in (1) and the operator H in (2) are quite
ince valid even if both the operator A Finally we emphasize once again that the operator *A* in (1) and the operator *H* in (2) are quite general. All the above-mentioned formulae are valid even if both the operator *A* and the operator *H* do not contain any linear terms. This means that the newly proposed boundary element method is still valid no matter whether or not there exist linear terms in the original governing equation (1) and boundary condition (2). Therefore it is possible for us to use the newly proposed boundary element method to solve more non-linear problems with strong non-linearity.

3. NUMERICAL EXAMPLES

In References 7 and 8, we showed that the proposed general BEM is valid for quite highly non-linear problems. However, the examples given in References 7 and 8 are only one-dimensional and are generally considered to be not satisfactory. Hence in this paper we apply the proposed general BEM to solve some 2D non-linear problems in order to show that the general BEM is indeed valid for highdimensional non-linear problems.

Example 1

Consider the 2D second-order non-linear differential equation

$$
\frac{1}{4}[(u_{xx})^2 + (u_{yy})^2] + \frac{1}{2}u_xu_y = e^{-2(x+y)}, \quad x \in [0, 1], \quad y \in [0, 1],
$$
\n
$$
\text{u}(x, y) = e^{-y}, \quad x = 0, \quad y \in [0, 1], \tag{18}
$$

with boundary conditions

$$
u(x, y) = e^{-y}, \quad x = 0, \quad y \in [0, 1], \tag{18}
$$

$$
(17)
$$

\n
$$
u(x, y) = e^{-y}, \quad x = 0, \quad y \in [0, 1], \quad y \in [0, 1], \quad (17)
$$

\n
$$
u(x, y) = e^{-y}, \quad x = 0, \quad y \in [0, 1], \quad (18)
$$

\n
$$
\frac{\partial u(x, y)}{\partial n} = e^{-x}, \quad x \in [0, 1], \quad y = 0, \quad (19)
$$

\n
$$
(19)
$$

\n
$$
y + \frac{\partial u(x, y)}{\partial n} = 0, \quad x = 1, \quad y \in [0, 1], \quad (20)
$$

\n
$$
y = \frac{\partial u(x, y)}{\partial n} = 0, \quad x = 1, \quad y \in [0, 1], \quad y = 1.
$$

\n(21)

$$
+(u_{yy})^{2}] + \frac{1}{2}u_{x}u_{y} = e^{-2(x+y)}, \quad x \in [0, 1], \quad y \in [0, 1], \tag{17}
$$
\n
$$
u(x, y) = e^{-y}, \quad x = 0, \quad y \in [0, 1], \tag{18}
$$
\n
$$
\frac{\partial u(x, y)}{\partial n} = e^{-x}, \quad x \in [0, 1], \quad y = 0, \tag{19}
$$
\n
$$
u(x, y) + \frac{\partial u(x, y)}{\partial n} = 0, \quad x = 1, \quad y \in [0, 1], \tag{20}
$$
\n
$$
\frac{\partial u(x, y)}{\partial n} + \frac{\partial u(x, y)}{\partial n} \cos[u(x, y)] = 0, \quad x \in [0, 1], \quad y = 1. \tag{21}
$$

$$
\frac{1}{4}[(u_{xx})^2 + (u_{yy})^2] + \frac{1}{2}u_xu_y = e^{-2(x+y)}, \quad x \in [0, 1], \quad y \in [0, 1],
$$
\n(17)

\nconditions

\n
$$
u(x, y) = e^{-y}, \quad x = 0, \quad y \in [0, 1],
$$
\n
$$
\frac{\partial u(x, y)}{\partial n} = e^{-x}, \quad x \in [0, 1], \quad y = 0,
$$
\n
$$
u(x, y) + \frac{\partial u(x, y)}{\partial n} = 0, \quad x = 1, \quad y \in [0, 1],
$$
\n
$$
u(x, y) \cos\left(\frac{\partial u(x, y)}{\partial n}\right) + \frac{\partial u(x, y)}{\partial n} \cos[u(x, y)] = 0, \quad x \in [0, 1], \quad y = 1.
$$
\n(21)

\nnphasized that the governing equation (17) does not contain any linear terms at all so onal BEM is invalid. Note that (18) is a Dirichlet-type condition, (19) is a Neumann-

 $\frac{(x, y)}{\partial n}$
 $x + \frac{\partial u}{\partial n}$
 $y + \frac{\partial v}{\partial n}$
 $y = 0$
 $y = c$
 $z = c$ e^{-x}
 $\frac{(x, y)}{\partial n}$
 $\frac{\partial u(x, y)}{\partial n}$ $z \in [0, 1],$ $y = 0,$ (19)
 $x = 1,$ $y \in [0, 1],$ (20)
 $y(x, y) = 0,$ $x \in [0, 1],$ $y = 1.$ (21)
 $y = 1$ $(x, y) + \frac{\partial u(x, y)}{\partial n}$
 $\frac{(x, y)}{\partial n}$ + $\frac{\partial u(x, y)}{\partial n}$

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vol. 24: 863-873 (19 = 0, $x = 1$, $y \in [0, 1]$, (20)
 $\frac{1}{2} \cos[u(x, y)] = 0$, $x \in [0, 1]$, $y = 1$. (21)

quation (17) does not contain any linear terms at all so

(18) is a Dirichlet-type condition, (19) is a Neumann-

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dition and $[u(x, y)] = 0, \quad x \in [0, 1], \quad y = 1.$ (21)
on (17) does not contain any linear terms at all so
is a Dirichlet-type condition, (19) is a Neumann-
d (21) is a condition which does not contain any
is one of the solutions of the abov It should be emphasized that the governing equation (17) does not contain any linear terms at all so that the traditional BEM is invalid. Note that (18) is a Dirichlet-type condition, (19) is a Neumanntype condition, (20) is a mixed-type condition and (21) is a condition which does not contain any linear terms. Note also that $u(x, y) = \exp(-x - y)$ is one of the solutions of the above problem so that $(x, y) = \exp(-x - y)$ is one of the solutions of the above problem so that
cal approximation with it.
VOL. 24: 863-873 (1997)
 a 1997 by John Wiley & Sons, Ltd we can compare our numerical approximation with it.

INT. J. NUMER. METHODS FLUIDS, VOL. 24: 863–873 (1997) $\qquad \qquad \odot$ 1997 by John Wiley & Sons, Ltd.

For all examples considered in this paper, we use the 2D Laplace operator

$$
L(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}
$$
 (22)

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 (y_j)) – e⁻⁽³ $\frac{\partial x^2}{\partial \Omega}$
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 (23) as the linear operator. For simplicity we always use u_0 $(x, y) = 0$ as the initial approximation. For the
 $[0, 1] \times [0, 1]$ into $N_{\Omega} \times N_{\Omega}$ equal subdomains

the unknowns are linearly distributed. At each

es are used to treat the discontinuity of the
 $xN_{\Omega} = N_{\Gamma} = 40$. In numerical domain integral we divide the domain $\Omega = [0, 1] \times [0, 1]$ into $N_{\Omega} \times N_{\Omega}$ equal subdomains
and each boundary into N_{Ω} equal elements in which the unknowns are linearly distributed. At each = [0, 1] ×
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 $\frac{(x_i + y_j)^2}{2}$
 $\left(\frac{(x_i, y_j)^2}{2}\right)$ [0, 1] into $N_{\Omega} \times$
owns are linearly
sed to treat the $N_{\Gamma} = 40$. In oray working two kind:
 $\left(x_i = \frac{i}{N_{\Omega}}, y_j = \frac{i}{N_{\Omega}}, y_j = \frac{i}{N_{\Omega}}\right)$ and each boundary into N_{Γ} equal elements in which the unknowns are linearly distributed. At each corner, two very close points on different boundaries are used to treat the discontinuity of the corner, two very close points on different boundaries are used to treat the discontinuity of the unknowns at corners. Throughout this paper we use $N_{\Omega} = N_{\Gamma} = 40$. In order to check whether iteration procedures converge or not, we investigate the following two kinds of root-mean-square errors:

orners. Throughout this paper we use
$$
N_{\Omega} = N_{\Gamma} = 40
$$
. In order to check whether

\nures converge or not, we investigate the following two kinds of root-mean-square

\n
$$
RMS_1 = \sqrt{\left(\sum_{i=0}^{N_{\Omega}} \sum_{j=0}^{N_{\Omega}} \frac{[u_k(x_i, y_j) - e^{-(x_i + y_j)}]^2}{(1 + N_{\Omega})^2}\right)} \quad \left(x_i = \frac{i}{N_{\Omega}}, y_j = \frac{j}{N_{\Omega}}\right),
$$
\n2MS_2 = \sqrt{\left(\sum_{i=0}^{N_{\Omega}} \sum_{j=0}^{N_{\Omega}} \frac{[u_{k+1}(x_i, y_j) - u_k(x_i, y_j)]^2}{(1 + N_{\Omega})^2}\right)} \quad \left(x_i = \frac{i}{N_{\Omega}}, y_j = \frac{j}{N_{\Omega}}\right),\n24

$$
RMS_{1} = \sqrt{\left(\sum_{i=0}^{N_{\Omega}} \sum_{j=0}^{N_{\Omega}} \frac{[u_{k}(x_{i}, y_{j}) - e^{-(x_{i}+y_{j})}]^{2}}{(1+N_{\Omega})^{2}}\right)} \left(x_{i} = \frac{i}{N_{\Omega}}, y_{j} = \frac{j}{N_{\Omega}}\right),
$$
(23)

$$
RMS_{2} = \sqrt{\left(\sum_{i=0}^{N_{\Omega}} \sum_{j=0}^{N_{\Omega}} \frac{[u_{k+1}(x_{i}, y_{j}) - u_{k}(x_{i}, y_{j})]^{2}}{(1+N_{\Omega})^{2}}\right)} \left(x_{i} = \frac{i}{N_{\Omega}}, y_{j} = \frac{j}{N_{\Omega}}\right),
$$
(24)

Figure 1. Errors of solution (RMS₁ defined by (23)) versus iterative time for Examples 1-3

Figure 2. Errors of solution $(RMS₂$ defined by (24)) versus iterative time for Examples 1–3

We simply apply the first-order ($M = 1$) expression (5) as our iterative formula and use $\lambda = 1$ as = 1) expression (5) as our iterative formula and use $\lambda = 1$ as
onverges quickly to the exact solution $\exp(-x - y)$, as shown
ze that the governing equation (17) does *not* contain any linear
valid for it. However, the propo the iterative parameter. The iteration converges quickly to the exact solution $exp(-x - y)$, as shown $(-x - y)$, as shown
t contain any linear
1 BEM works quite
ensional non-linear
with more complex
(0, 1], (25)
 -0 (26) in Figures 1 and 2. We should emphasize that the governing equation (17) does *not* contain any linear terms so that the traditional BEM is invalid for it. However, the proposed general BEM works quite well. This example illustrates that the general BEM is indeed valid for high-dimensional non-linear problems whose governing equation does not contain any linear terms.

Example 2

In the second example we consider again the same governing equation (17) but with more complex non-linear boundary conditions

$$
u\cos\left(\frac{\partial u}{\partial n}\right) - \frac{\partial u}{\partial n}\cos(u) + \sin^2(u) + \cos^2\left(\frac{\partial u}{\partial n}\right) = 1, \quad x = 0, \quad y \in [0, 1],
$$
(25)

$$
u\cos\left(\frac{\partial u}{\partial n}\right) - \frac{\partial u}{\partial n}\cos(u) + \sin^2(u) + \cos^2\left(\frac{\partial u}{\partial n}\right) = 1, \quad x \in [0, 1], \quad y = 0,
$$
(26)

$$
\int \frac{\partial u}{\partial n} \cos\left(\frac{\partial u}{\partial n}\right) - \frac{\partial u}{\partial n} \cos(u) + \sin^2(u) + \cos^2\left(\frac{\partial u}{\partial n}\right) = 1, \quad x \in [0, 1], \quad y = 0,
$$
\n
$$
u \cos\left(\frac{\partial u}{\partial n}\right) + \frac{\partial u}{\partial n} \cos(u) + u \frac{\partial u}{\partial n} + e^{-2(1+x)} = 0, \quad x \in [0, 1], \quad y = 1,
$$
\n(27)

$$
\begin{aligned}\n\int \frac{\partial u}{\partial n} \cos(u) + \sin(u) + \cos\left(\frac{\partial u}{\partial n}\right) &= 1, \quad x \in [0, 1], \quad y = 0, \\
u \cos\left(\frac{\partial u}{\partial n}\right) + \frac{\partial u}{\partial n} \cos(u) + u \frac{\partial u}{\partial n} + e^{-2(1+x)} &= 0, \quad x \in [0, 1], \quad y = 1,\n\end{aligned} \tag{27}
$$

$$
u\cos\left(\frac{\partial u}{\partial n}\right) + \frac{\partial u}{\partial n}\cos(u) + u\frac{\partial u}{\partial n} + e^{-2(1+x)} = 0, \quad x \in [0, 1], \quad y = 1,
$$
 (27)

$$
u\cos\left(\frac{\partial u}{\partial n}\right) + \frac{\partial u}{\partial n}\cos(u) + u\frac{\partial u}{\partial n} + e^{-2(1+y)} = 0, \quad x = 1, \quad y \in [0, 1].
$$
 (28)
nphasize that both the governing equation (17) and all four boundary conditions (25)-

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 $(u) + \cos^2\left(\frac{\partial u}{\partial n}\right)$
 $\frac{u}{n} + e^{-2(1+x)} = 0$
 $\therefore + e^{-2(1+y)} = 0$
 \therefore the called that *u*
 \therefore pression (5) as = 1, *x* = 0, *y* ∈ [0, 1], (25)

= 1, *x* ∈ [0, 1], *y* = 0, (26)

, *x* ∈ [0, 1], *y* = 1, (27)
 x = 1, *y* ∈ [0, 1]. (28)

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, *x* ∈ [0, 1], *y* = 1, (27)
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 $u' = 1$ e obtain
 $u' = 1$ e obtain $+e^{-2(1+x)} = 0$, $x \in [0, 1]$, $y = 1$, (27)
 $-e^{-2(1+y)} = 0$, $x = 1$, $y \in [0, 1]$. (28)

g equation (17) and all four boundary conditions (25)—

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b *u u* $+e^{-2(1+y)} = 0$, $x = 1$, $y \in [0, 1]$. (28)

ong equation (17) and all four boundary conditions (25)–

ote also that $u(x, y) = \exp(-x - y)$ is one of the solutions

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1 nu $(x, y) = \exp(-x - y)$ is one of the solutions
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tai We simply use the first-order ($M = 1$) expression (5) as our iterative formula and use $\lambda = 1$ as the = 1) expression (5) as our iterative formula and use $\lambda = 1$ as the
obtain a numerical result which converges to the exact solution
rors versus iterative times are shown in Figures 1 and 2. This
general BEM is valid even iterative parameter. Once again we obtain a numerical result which converges to the exact solution $\exp(-x - y)$. The corresponding errors versus iterative times are shown in Figures 1 and 2. This example indicates that the proposed general BEM is valid even for those non-linear problems whose governing equations and boundary conditions do *not* contain any linear terms at all!

Example 3

Finally, we consider the more complex non-linear differential equation

$$
(-x - y)
$$
. The corresponding errors versus iterative times are shown in Figures 1 and 2. This
mple indicates that the proposed general BEM is valid even for those non-linear problems whose
erning equations and boundary conditions do *not* contain any linear terms at all!
mple 3
Finally, we consider the more complex non-linear differential equation

$$
\frac{1}{4}[(u_{xx})^2 + (u_{yy})^2] + \frac{1}{2}u_xu_y + \ln\left(\frac{1 + \sin^2(u_{xx}u_{yy}) + \cos^2(u_xu_y)}{2}\right) = \alpha e^{-2(x+y)},
$$

$$
x \in [0, 1], \quad y \in [0, 1], \alpha \in (0, \infty),
$$
(29)
in the same boundary conditions (25)–(28). Note that both the governing equation and boundary
ditions do not contain any linear terms. In particular the governing equation (29) contains

with the same boundary conditions (25) – (28) . Note that both the governing equation and boundary conditions do not contain any linear terms. In particular the governing equation (29) contains trigonometric and logarithmic functions of the non-linear expressions of the unknown $u(x, y)$, so that the non-linearity of the governing equation (29) is quite high.

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(25)–(28). Note that both the governing equation and boundary
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This example
viley & Sons, Ltd In the case $\alpha = 1$, the real function $\exp(-x - y)$ is one of the solutions of the third example. In the = 1, the real function exp($-x$ *y*) is one of the solutions of the third example. In the
simply apply the first-order (*M* = 1) expression (5) as our iterative formula and use
erative parameter. Once again, we successfu case $\alpha = 1$ we simply apply the first-order $(M = 1)$ expression (5) as our iterative formula and use = 1 we simply apply the first-order (*M* = 1) expression (5) as our iterative formula and use
as the iterative parameter. Once again, we successfully obtain a numerical result which
ges to the exact solution exp($-x - y$), a $\lambda = 1$ as the iterative parameter. Once again, we successfully obtain a numerical result which = 1 as the iterative parameter. Once again, we successfully obtain a numerical result which
noverges to the exact solution $exp(-x - y)$, as shown in Figures 1 and 2. The numerical results of *u*
 $d \frac{\partial u}{\partial n}$ on the four boun converges to the exact solution $exp(-x - y)$, as shown in Figures 1 and 2. The numerical results of u $(-x - y)$, as shown in Figures 1 and 2. The numerical results of *u* gree quite well with the corresponding values of the exact solution, hat we use $u_0(x, y) = 0$ as the initial approximation. This example (863–873 (1997) (2 and $\partial u/\partial n$ on the four boundaries agree quite well with the corresponding values of the exact solution, $\partial u/\partial n$ on the four boundaries agree quite well with the corresponding values of the exact solution,
nown in Figures 3 and 4. Note that we use $u_0(x, y) = 0$ as the initial approximation. This example
0. NUMER. METHODS FLU as shown in Figures 3 and 4. Note that we use u_0 $(x, y) = 0$ as the initial approximation. This example
 $\circled{ }$ 1997 by John Wiley & Sons, Ltd

Figure 3. Comparison of numerical results on boundaries with corresponding exact solution of Example 3 in case $\alpha = 1$: curve 1, *u* on boundary $x \in [0, 1]$, $y = 1$; curve 2, *u* on boundary $x \in [0, 1]$, $y = 0$; curve 3 $\partial u / \partial n$ on boundary $x \in [0, 1]$, $y = 0$; curve 4, $-\frac{\partial u}{\partial n}$ on boundary $x \in [0, 1]$, $y = 0$; crosses, corresponding values of exact solution

indicates that the proposed general BEM is valid even for quite complex, highly non-linear differential equations with quite complex non-linear boundary conditions. Note that both the governing equation and boundary conditions of the third example do not contain any linear terms.

In the case $\alpha \neq 1$ we also obtain convergent numerical results which are different from $\exp(-x - y)$, as shown in Figures 5–8. The numerical parameters for the third example are given in Table I. We apply both the first-order $(M = 1)$ and second-order $(M = 2)$ iterative formulae of (5). We find that if the second-order iterative formula ($M = 2$) converges under a value of $\lambda = \mu$, then it = 1: curve
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,, then it ϵ [0, 1], *y* = 1; curve 2, *u* on boundary *x* ϵ [0, 1], *y* = 0; curve 3 $\partial u/\partial n$ on boundary *x* ϵ [0, 1], *y* = 0; crosses, corresponding values of exact solution the proposed general BEM is valid even for qu $-\frac{\partial u}{\partial n}$ on boundary $x \in [0, 1], y = 0$; crosses, corresponding values of exact solution
ne proposed general BEM is valid even for quite complex, hightions with quite complex non-linear boundary conditions. Note
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Figure 4. Comparison of numerical results on boundaries with corresponding exact solution of Example 3 in case $\alpha = 1$: curve = 1: curve
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= 373 (1997) 1, *u* on boundary $y \in [0, 1]$, $x = 1$; curve 2, *u* on boundary $y \in [0, 1]$, $x = 0$; curve 3 $\partial u/\partial n$ on boundary $x \in [0, 1]$, $x = 1$; curve $\epsilon \in [0, 1], x = 1$; curve 2, *u* on boundary $y \in [0, 1], x = 0$; curve 3 $\partial u / \partial n$ on boundary $x \in [0, 1], x = 1$; curve, $-\partial u / \partial n$ on boundary $y \in [0, 1], x = 0$; crosses, corresponding values of exact solution $\exists y \&$ Sons, Ltd. 4, $-\frac{\partial u}{\partial n}$ on boundary $y \in [0, 1]$, $x = 0$; crosses, corresponding values of exact solution $-\frac{\partial u}{\partial n}$ on boundary $y \in [0, 1], x = 0$; crosses, corresponding values of exact solution & Sons, Ltd.

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Figure 5. Solution of Example 3 in case $\alpha = 0.5$

converges faster than by the first-order formula ($M = 1$) under the same value of $\lambda = \mu$. Moreover, if the iteration converges under a value of $\lambda = \mu_1$, then the same iterative formula under a value of $\lambda = \mu_2 < \mu_1$ converges more slowly. In Reference 8, we show similar results for one-dimensional $\lambda - \mu_2$ \le examples.

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amples.
It should be emphasized that only *one* linear operator, i.e. the 2D Laplace operator, and its
corresponding fundame It should be emphasized that only *one* linear operator, i.e. the 2D Laplace operator, and its corresponding fundamental solution are used for *all* the above quite different non-linear problems. This is very interesting and deserves further research in detail. It seems that a fairly general BEM computer programme might be developed to solve a large number of quite different sorts of highdimensional strongly non-linear problems in engineering, especially when the proposed general BEM is combined with the well-established dual reciprocity method that can transform the domain integral into the surface so that much less CPU capacity is necessary. $= 0.5$
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Figure 6. Solution of Example 3 in case $\alpha = 10$ $=10$

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Figure 7. Solution of Example 3 in case $\alpha = 100$

Figure 8. Solution of Example 3 in case $\alpha = 250$ $= 250$

4. CONCLUSIONS AND DISCUSSION

In this paper the general boundary element method for strongly non-linear problems proposed by Liao^{7,8} is proved to be valid for quite complex, strongly non-linear 2D differential equations, even including those whose governing equations and boundary conditions do *not* contain any linear terms at all. Based on these examples, we have many reasons to believe that the proposed general BEM can be applied to solve high-dimensional strongly non-linear problems in engineering. Note that the proposed general BEM has been successfully applied to solve viscous flows governed by the N–S equations, $\frac{5,10}{ }$ the non-linear heat transfer of inhomogeneous materials, etc.

It should be emphasized that we use the 2D Laplace operator as the linear operator for all three quite different non-linear problems under consideration. This is very interesting. It implies that a general BEM software for different types of strongly non-linear problems might be developed, because a simple linear operator whose fundamental solution is known might be used for a large number of quite different types of non-linear differential equations, as illustrated in this paper.

From the theoretical viewpoint it seems that the proposed general BEM might overcome nearly all restrictions of the traditional BEM for non-linear problems and could be applied to solve reasonable non-linear differential equations. However, as in the traditional BEM for non-linear problems, the domain integral term appears in the general BEM, which decreases greatly the effectiveness of the proposed general BEM. There might exist two ways to overcome this disadvantage of the proposed general BEM. One is to use a vector supercomputer, because the parallel process is especially simple and quite effective for the integral. The other is to apply the so-called dual reciprocity method¹¹ which was developed to increase the effectiveness of the traditional BEM for non-linear problems by means of transforming the domain integral onto the surface. Both deserve further researches in detail.

Finally we would like to point out that the general BEM proposed in References 7 and 8 has a solid mathematical base. In fact, it is only a simple application of a newly proposed non-linear analytical technique, the homotopy analysis method, $1^{2,13}$ which is based on homotopy in topology and has been successfully applied to solve many non-linear problems. The author has even applied the homotopy analysis method to obtain some wonderful results in pure mathematics, such as the generalized Newtonian binomial theorem about $(1 + t)^a$ for fractional and negative exponents which has been $(1 + t)$
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